International Journal of Scientific & Engineering Research, Volume 7, Issue 11, November-2016 401
ISSN 2229-5518

Approximating Solution for Quadratic Functional Integral Equation with Maxima

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Abstract— In this paper we prove an existence as well as approximation result for nonlinear quadratic functional integral equation with maxima .An algorithm for the solution is developed and it is shown that the sequence of successive approximation starting with a lower or an upper solution converges monotonically to the solution of related quadratic functional integral equation with maxima under some suitable mixed hybrid conditions.We base our main results on the Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage 2014 in a partially ordered normed linear space. An example illustrating the existence result is also presented.

Index Terms— Quadratic functional integral equation with maxima, hybrid fixed point theorem, approximate solution, iteration method

1 Introduction

The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe .See Argyros[2], Deimling [6], Chandrasekher [4] and the references therin. The study gained momentum after the formulation of fixed point principles in Banach algebras due to Dhage [12], [14]. The existence results for such equations are generally proved uder the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities of the quadratic integral equations. See Dhage [12], [14] and the references therein. The Lipschitz and compactness hypoteses are considered to be very strong conditions in the theory of nonlinear differential and integral equations which do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations.However, the literature on existence results for a special class of functional differential equations, namely nonlinear quadratic differential equations with maxima under weaker partial Lipschitz and partial compactness type conditions via Dhage iteration method is not enriched yet, recently, the first authors in [14] ,[15],[16] have studied the existence results of functional differential equations with maxima. Therefore, it is admirable to extend this method to nonlinear quadratic integral equations with maxima. This is the main motivation of the present paper.

In this paper we prove the existence as well as approximations of the positive solutionsof a certain quadratic integral equation with maxima via an algorithm based on successive approximations under partially Lipschitz and compactness conditions.

Given a closed and bounded interval J = [0, T] of the real line R for some T > 0, we consider the quadratic functional integral equation (in short QFIE) with maxima

$$x(t) = f[t, x(t), X(t)]\left(q(t) + \int_0^t g(s, x(s), X(s)ds\right) \tag{1}$$

for all t ∈ J, where the functions f, g : J × R × R → R, are continuous functions, and $X(t) = \max_{0\le\eta\le t} x(\eta)$.

By a solution of the QFIE (1.1) with maxima we mean a function x ∈ C (J, R) that satisfies the equation (1.1) on J, where C(J, R) is the space of continuous real-valued functions defined on J.

The QFIE (1.1) with maxima is new to the literature. In particular, If g (t, x, y) = 0 for all t ∈ J and x, y ∈ R the QFIE (1.1) with maxima reduces to the nonlinear functional equation with maxima

$$x(t) = f\big(t, x(t), X(t)\big),\ t \in J \tag{2}$$

and if f (t, x, y) = 1 for all t ∈ J and x, y ∈ R, it is reduced to nonlinear usual Volterra integral equation with maxima

$$x(t) = \ q(t) + \int_0^t g(s, x(s), X(s)ds \tag{3}$$

Therefore, the QFIE (1) is general and the results of this paper include the existence and approximations results for above nonlinear functional and Volterra integral equations with maxima as special cases.

2 Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation ≤ and the norm ‖.‖ in which the addition and thescalar multiplication by positive real numbers are preserved by ≤ .

A few details of a partially ordered normed linear space appear in Dhage [10], Heikkila and Lakshmikantham [18] and the references therein.
We need the following notion and results.

Definition 2.1.

A mapping $T : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\leq$ that is, if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in E$. Similarly, T is called monotone nonincreasing if $x \geq y$ implies $Tx \geq Ty$ for all $x; y \in E$. Finally, T is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2.

A mapping $T : E \to E$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|T_x - T_y\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. T called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3.

A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator T on a partially normed linear space E into itself is called partially bounded if T (E) is a partially bounded subset of E. T is called uniformly partially bounded if all chains C in T (E) are bounded by a unique constant.

Definition 2.4.

A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is a relatively compact subset of E. A mapping $T : E \to E$ is called partially compact if T (E) is a partially relatively compact subset of E. T is called uniformly partially compact if T is a uniformly partially bounded and partially compact operator on E. T is called partially totally bounded if for any bounded subset S of E, T (S) is a partially relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 2.5.

Suppose that T is a nondecreasing operator on E into itself. Then T is a partially bounded or partially compact if T (C) is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.6.

The order relation $\leq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}_{n\in N}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{nk}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in N}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leq, \|.\|)$ the order relation $\leq$ and the norm $\|.\|$ are said to be compatible if $\leq$ and the metric d defined through the norm $\|.\|$ are compatible.
Clearly, the set R of real numbers with usual order relation $\leq$ and the norm $\|.\|$ defined by the absolute value function $|.|$ has this property. Similarly, the finite dimensional Euclidean space R^n with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.7.

A upper semi-continuous and monotone nondecreasing function $\psi: R_+ \to R_+$ is called a D-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \leq, \|.\|)$ be a partially ordered normed linear space. A mapping $T : E \to E$ is called partially nonlinear D-Lipschitz if there exists a D-function $\psi: R_+ \to R_+$ such that

$$\|T_x - T_y\| \leq \psi(\|x - y\| \quad (3)$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then T is called a partially Lipschitzwith a Lipschitz constant k.
Let $(E, \leq, \|.\|)$ be a partially ordered normed linear algebra. Denote
$E+ = \{x \in E \mid x \geq \theta$, where θ is the zero element of E }
And
$K = \{E^+ \subset E | uv \in E^+ \text{for all } u, v \in E^+\}$ (4)
The elements of K are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set K and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Remark 2.8.(Dhage [10])

If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leq v_1$ and $u_2 \leq v_2$, then $u_1 u_2 \leq v_1 v_2$.

Definition 2.9.

An operator $T : E \to E$ is said to be positive if the range R(T) of T is such that $R(T) \subset K$.

Theorem 2.10(Dhage [10])

Let $(E, \leq, \|.\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\leq$ and the norm $\|.\|$ in E are compatible in every compact chain of E. Let $A, B: E \to E$ be two nondecreasing operators such that
(a) A is partially bounded and partially nonlinear D-Lipschitz with D-functions ψ_A.
(b) B is partially continuous and uniformly partially compact.
(c)$M\,\psi_A(r) < r, r > 0$,where
$M = \sup\{\|B(C)\| : C$ is a chain in E} and
(d) there exists an element $x_0 \in X$ such that $x_0 \leq Ax_0\, Bx_0$ or $x_0 \geq Ax_0\, Bx_0$.

Then the operator equation

$$Ax\,Bx = x \quad (5)$$

has a solution x^* in E and the sequence $\{x_n\}_{n\in N}$ of successive iterations defined by $x_{n+1} = Ax_n Bx_n$, n = 0,1,……. converges monotonically to x^*.

Remark 2.11.

The compatibility of the order relation ≤ and the norm ‖.‖ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to ≤ and ‖.‖. This simple fact has been utilized to prove the main results of this paper.

3 Existence and Approximation Result

The QFIE (1) is considered in the function space $C(J,R)$ of continuous real-valued functions
defined on J. We define a norm ‖.‖ and the order relation ≤ in $C(J,R)$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{6}$$

And

$$x \le y \Rightarrow x(t) \le y(t) \quad \forall\, t \in J. \tag{7}$$

respectively. Clearly, C(J;R) is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation ≤. It is known that the partially ordered Banach algebra C(J;R) has some nice properties concerning the compatibility property with respect to the norm ‖.‖ and the order relation ≤ in certain subsets of of it. The following lemma in this connection follows by an application of Arzella-Ascoli theorem.

Lemma 3.1.

Let (C(J, R) , ≤, ‖.‖) be a partially ordered Banach space with the norm ‖.‖ and the order relation ≤ defined by (6) and (7) respectively. Then every partially compact subset S of $(J;R)$, ‖.‖ and ≤ are compatible in every compact chain C in S.

Proof. Let S be a partially compact subset of C(J;R) and let $\{x_n\}_{n\in N}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le \ x_2(t) \le \_\_\_\_\_ x_n(t) \tag{8}$$

for each $t \in J$. Suppose that a subsequence $\{x_{nk}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{nk}\}_{n\in N}$ of the monotone real sequence $\{x_n\}_{n\in N}$ is convergent. By monotone characterization, the whole sequence $\{x_n\}_{n\in N}$ is convergent and converges to a point x(t) in R for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in N}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence

$\{x_n\}_{n\in N}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in N}$ is an equicontinuous sequence by Arzel_a-Ascoli theorem. Hence $\{x_n\}_{n\in N}$ is convergent and converges uniformly to x. As a result ‖.‖ and ≤ are compatible in S. This completes the proof.
We need the following definition in what follows

Definition 3.2.

A function $u \in C(J;R)$ is said to be a lower solution of the QFIE (1) if it satisfies

$$x(t) = f[t, u(t), U(t)]\left(q(t) + \int_0^t g(s, u(s), U(s)ds\right)$$

for all $t \in J$. Similarly, a function $v \in C(J;R)$ is said to be an upper solution of the QFIE (1) with maxima if it satisfies the above inequalities with reverse sign.
We consider the following set of assumptions in what follows
(A_1) q defines a continuous function $q : J \to R_+$
(A_2) The function f is nonnegative on $f : J \times R \times R \to R$
(A_3) There exists a D-function ψ_f such that

$$0 \le f(t, x_1, x_2) - f(t, y_1, y_2) \le \psi_f(max\{x_1 - y_1, x_2 - y_2\})$$

for all t ∈ J and x_1, x_2, y_2, y_2 ∈ R, $x_1 \ge y_1$, $x_2 \ge y_2$.
(A_4) There exists a constant M_f such that 0 ≤ f (t, x, y) ≤ M_f for all t ∈ J and x, y ∈ R.
(A_5) g (t, x, y) is nondecreasing in x and y for all t ∈ J.
(A_6) There exists a constant M_g > 0 such that g (t, x, y) ≤ M_g for all t ∈ J and x, y ∈ R.
(A_7) The QFIE (1) with maxima has a lower solution u ∈ C (J, R).

Theorem3.3.

Assume that hypotheses (A_1) - (A_7) hold. Furthermore, assume that

$$(q + T M_g)\psi_f(r) < r, r > 0, \tag{9}$$

then the QFIE (1) with maxima has a solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ of successive approximations defined by

$$x_{n+1}(t) = f[t, x_n(t), X_n(t)]\left(q(t) + \int_0^t g(s, x_n(s), X_n(s)ds\right)$$

for all t ∈ J, where x_0 = u and $X_n(t) = \max_{0\le\eta\le t} x_n(\eta)$, converges monotonically to x^* .

Proof. Set E = C(J;R). Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm ‖.‖ and the order relation ≤ in E.
Define two operators A and B on E by

$$Ax(t) = f(t, x(t), X(t)),\ t \in J \tag{10}$$

and

$$Bx(t) = q(t) + \int_0^t g(s, x(s), X(s)ds \tag{10}$$

From the continuity of the integral and the hypotheses (A1)-(A_6), it follows that A and B define the maps $A, B: E \rightarrow E$. Now by definitions of the operators A and B, the QFIE (1) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{11}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 2.10. This is achieved in the series of following steps

Step I: A and B are nondecreasing on E.

Let x, y ∈ E be such that x ≥ y. Then x (t) ≥ y(t) for all t ∈ J. Since y is continuous on [a, t], there exists a η^* ∈ [a, t] such that y (η^*) = max y(η). By definition of ≤, one has x (η^*) ≥ y (η^*) Consequently, we obtain

$X(t) = \max_{0 \le \eta \le t} x(\eta) = x(\eta^*) \ge y(\eta^*) = \max_{0 \le \eta \le t} y(\eta) = Y(t)$

for each t ∈ J. Then by hypothesis (A_2), we obtain

$$A\,x(t) = f(t, x(t), X(t)) \ge f(t, y(t), Y(t))) = Ay(t)$$

for all t ∈ J. This shows that A is nondecreasing operators on E into E. Similarly, using hypothesis (A_5),

$$Bx(t) = q(t) + \int_0^t g(s, x(s), X(s)ds$$
$$\ge q(t) + \int_0^t g(s, x(s), Y(s)ds$$
$$= By(t)$$

for all t ∈ J. Hence, it is follows that the operator B is also a nondecreasing operator on E into itself. Thus, A and B are nondecreasing positive operators on E into itself

Step II: A is partially bounded and partially D-Lipschitz on E.

Let x ∈ E be arbitrary. Then by (A_2),

$$|Ax(t)| \le f(t, x(t), X(t)) \le Mf,$$

for all t ∈ J. Taking supremum over t, we obtain Ax ≤ Mf and so, A is bounded. This further implies that A is partially bounded on E. Next, let $x, y \in E$ be such that $x \ge y$. Then, we have

$$|x(t) - y(t)| \le |X(t) - Y(t)|$$

and

$$|X(t) - Y(t)| = X(t) - Y(t)$$
$$= \max_{t_0 \le \eta \le t} x(\eta) - \max_{t_0 \le \eta \le t} y(\eta)$$
$$= \max_{t_0 \le \eta \le t} [x(\eta) - y(\eta)]$$
$$\le \|x - y\|$$

for each t ∈ J. As a result, by hypothesis (A_3),

$$|Ax(t) - Ay(t)| = |f(t, x(t), X(t)) - f(t, y(t), Y(t))|$$
$$\le \psi_f(\max\{|x(t) - y(t)|, |X(t) - Y(t)|\})$$
$$\le \psi_f(\|x - y\|)$$

for all t ∈ J. Taking supremum over t, we obtain

$$\|Ax - Ay\| \le \psi f(\|x - y\|)$$

for all x, y ∈ E with x ≥ y. Hence A partially nonlinear D-Lipschitz operators on E which further implies that A partially continuous on E.

Step III: B is a partially continuous operator on E.

Let $\{x_n\}_{n \in N}$ be a sequence in a chain C of E such that $x_n \rightarrow x$ for all n ∈ N. Then, by dominated convergence theorem, we have

$$\lim_{x\to\infty} Bx_n(t) = \lim_{x\to\infty} q(t) + \int_0^t g(s, x_n(s), X_n(s)ds$$
$$= q(t) + \int_0^t \lim_{x\to\infty} [g(s, x_n(s), X_n(s)]ds$$
$$= q(t) + \int_0^t g(s, x_n(s), X_n(s)ds$$
$$= Bx(t)$$

for all t ∈ J. This shows that Bxn converges monotonically to Bx pointwise on J. Next, we will show that $\{Bx_n\}_{n \in N}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$|Bx_n(t_2) - Bx_n(t_1)| = |q(t_2) - q(t_1)| + \left|\int_0^{t_2} g(s, x(s), X(s)ds - \int_0^{t_1} g(s, x(s), X(s)ds\right|$$
$$= |q(t_2) - q(t_1)| + \left|\int_{t_1}^{t_2} |g(s, x(s), X(s)|ds\right|$$
$$\le |q(t_2) - q(t_1)| + M_g|t_2 - t_1|$$
$$\rightarrow 0, \quad t_2 - t_1 \rightarrow 0$$

uniformly for all n ∈ N. This shows that the convergence $Bx_n \rightarrow Bx$ is uniform and hence B is partially continuous on E.

Step IV: B is uniformly partially compact operator on E.

Let C be an arbitrary chain in E. We show that B(C) is a uniformly bounded and equi-continuous set in E. First we show that B(C) is uniformly bounded. Let y ∈ B(C) be any element Then there is an element x ∈ C be such that y = Bx. Now, by hypothesis (A_5),

$$y(t) = q(t) + \int_0^t |g(s, x(s), X(s)|ds$$
$$\le q + M_g T$$
$$\le r$$

for all t ∈ J. Taking the supremum over t, we obtain y ≤ Bx ≤ r for all y ∈ B(C). Hence, B(C) is a uniformly bounded subset of E. Moreover, B(C) ≤ r for all chains C in E. Hence B is a uniformly partially bounded operator on E Next, we will show that B(C) is an equicontinuous set in E. Let t_1, t_2 ∈ J be arbitrary with $t_1 < t_2$. Then, for any y ∈ B(C), one has

$$|y(t_2) - y(t_1)| = |Bx(t_2) - Bx(t_1)|$$

$$|q(t_1) - q(t_2)| + \left| \int_0^{t_2} g(s, x(s), X(s)ds - \int_0^{t_1} g(s, x(s), X(s)ds \right|$$

$$= |q(t_1) - q(t_2)| + \left| \int_{t_1}^{t_2} |g(s, x(s), X(s)|ds \right|$$

$$\leq |q(t_1) - q(t_2)| + M_g |t_2 - t_1|$$

$$\to 0 \ , t_2 - t_1 \to 0$$

uniformly for all y ∈ B(C). Hence B(C) is an equicontinuous subset of E. Now, B(C) is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, B is a uniformly partially compact operator on E into itself.

Step V : u satisfies the operator inequality u ≤ Au Bu.

By hypothesis (A_7), the QFIE (1) has a lower solution u defined on J. Then, we have

$$u(t) = f[t, u(t), U(t)] \left(q(t) + \int_0^t g(s, u(s), U(s)ds \right) \tag{12}$$

for all t ∈ J. From the definitions of the operators A and B it follows that u(t) ≤ Au(t) Bu(t) for all t ∈ J, hence u ≤ Au Bu.

Step VI : The D-functions ψA satisfy the growth condition M ψA (r) < r, for r > 0.Finally, the D-function ψA of the operator A satisfiy the inequality given in hypothesis (d) of Theorem 2.10, viz.,

$$M\, \psi A\,(r) \leq (q + Mg\, T)\, \psi f\,(r) < r$$

for all $r > 0$.

Thus A, B and C satisfy all the conditions of Theorem 2.10 and we conclude that the operator equation Ax Bx = x has a solution. Consequently the QFIE (1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in N}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof.

Example

Given a closed and bounded interval J = [0, 1], consider the QFIE,

$$\boldsymbol{x(t) = [2 + arctanX(t)] \left(\frac{1}{t+1} + \int_0^t \frac{[1+tanX(s)]}{4}\right)}$$

(13)

For t∈ J, where$X(t) = \max_{t_0 \leq \eta \leq t} x\,(\eta)$.

Here, $q(t) = \frac{t}{t+1}$ which is continuous and $q(t) = \frac{1}{2}$.Similarly, the functions f and g are defined by

$$f(t, x, y) = [2 + arctanx + arctany]$$

$$g(t, x, y) = g(t, x) = \frac{1 + tanhx}{4}$$

The function f satisfies the hypothesis (A_3) with $\psi f\ (r) = \frac{r}{1+\eta^2}$

For each $0 < \eta < r$. To see this, we have

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2)$$

$$\leq \frac{1}{1+\eta^2}(x_1 - y_1) + \frac{1}{1+\eta^2}(x_2 - y_2)$$

$$\leq \frac{1}{1+\eta^2} max\{x_1 - y_1, x_2 - y_2\}$$

for all x_1 , y_1 , x_2 , y_2 ∈ R, $x_1 \geq y_1$ and $x_2 > \xi > y_2$. Moreover, the function f is nonnegative and bounded on J × R × R with bound Mf = 3 and so the hypothesis (A_2) is satisfied.Again, since g is nonnegative and bounded on J ×R×R with bound Mg $= \frac{1}{2}$ the hypothesis(A_5) holds. Furthermore, g(t, x, y) = g(t, x) is nondecreasing in x and y for all t ∈ J, and thus hypothesis (A6) is satisfied.Thus, condition (9) of Theorem 3.3 is held. Finally, the QFIE (13) has a lower solution u (t) = 0 on J. Thus all the hypotheses of Theorem 3.3 are satisfied. Hence we apply Theorem 3.3 and conclude that the QFIE (13) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ defined by for all t ∈ J, where $x_0 = 0$, converges monotonically to x^*.

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